**Recall *u*-substitution technique:**

**Last Time:**

special case

$\int \sqrt{1+x^2}\, x\, dx$ $\quad u = 1+x^2$, $\frac{1}{2}du = x\,dx$

$= \int \sqrt{u} \cdot \frac{1}{2} du$

$= \frac{1}{2}\int u^{1/2}\, du$

$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3} u^{3/2} + C$

$= \boxed{\frac{1}{3}(1+x^2)^{3/2} + C}$

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$

$u^{1/2} u^1 = u^{1/2+1}$

versus

$\int \sqrt{1+x}\, x\, dx$ $\quad u = 1+x$, $du = dx$, $u-1 = x$

$= \int \sqrt{u}(u-1)\, du$

$= \int u^{1/2}(u-1)\, du$

$= \int (u^{3/2} - u^{1/2})\, du$

$= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C$

$\frac{3}{2} + \frac{2}{2} = \frac{5}{2}$

$= \boxed{\frac{2}{5}(1+x)^{5/2} - \frac{2}{3}(1+x)^{3/2} + C}$

Do: $\int e^{5x} dx$ ← use u-sub for both → Do: $\int \sin 9x\, dx$

$u = 5x$, $du = 5\,dx \Rightarrow \frac{1}{5}du = dx$

$= \frac{1}{5}\int e^u du$

$= \frac{1}{5} \cdot e^u + C$

$= \boxed{\frac{1}{5}e^{5x} + C}$

$u = 9x$, $du = 9\,dx \Rightarrow \frac{1}{9}du = dx$

$= \frac{1}{9}\int \sin u\, du$

$= -\frac{1}{9}\cos u + C$

$= \boxed{-\frac{1}{9}\cos 9x + C}$

$(\cos u)' = -\sin u$

$(-\cos u)' = \sin u$

Naturally, *u*-sub does not work in every situation.

**Recall: Derivative Product Rule**

$$[f \cdot g]' = f' \cdot g + f \cdot g'$$ ← let's integrate both sides

then $\int (fg)'\, dx = \int (f'g + fg')\, dx$

$fg = \int gf'\, dx + \int fg'\, dx$

$fg - \int gf'\, dx = \int fg'\, dx$

$$\boxed{uv - \int v\, du = \int u\, dv}$$

let $u = f$ $\quad v = g$

$du = f'dx$ $\quad dv = g'dx$

Integration by Parts (IBP)

**Another Tool: Integration by Parts (IBP)**

one function will become $u$

and the other function will be a derivative but NOT of $u$, label it $dv$

pick $u$ such that it will simplify / disappear as a derivative

ex. $\int \underset{u}{x}\underbrace{e^x dx}_{dv}$

BUILD A CHART:

$u = x \qquad dv = e^x dx$

$du = dx$ integrate / antiderivative

$v = e^x$ ← don't put a $+C$

$$\int u\,dv = uv - \int v\,du$$
$$= xe^x - \int e^x dx$$
↑ easy to integrate :)
$$= \boxed{xe^x - e^x + C}$$

**IBP Formula:** $\int u\,dv = uv - \int v\,du$

$\int u\ dv = uv - \int v\ du$

ex. $\int \underset{u}{x}\underbrace{e^{3x}dx}_{dv}$

$u = x \qquad dv = e^{3x}dx$

$du = dx \qquad v = \frac{1}{3}e^{3x}$

$u = 3x$

$du = 3dx \Rightarrow \frac{1}{3}du = dx$

$\int e^{3x}dx$

$= \frac{1}{3}\int e^u du$

$= \frac{1}{3}e^{3x}$

$$\int u\,dv = uv - \int v\,du$$
$$= x\left(\frac{1}{3}e^{3x}\right) - \frac{1}{3}\int e^{3x}dx$$
$$= \frac{1}{3}xe^{3x} + C_1 - \underbrace{\frac{1}{3}\cdot\frac{1}{3}}e^{3x} + C_2$$
$$= \boxed{\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C}$$

$C = C_1 + C_2$

$\int u\,dv$

ex. $\int x\sin x\,dx = uv - \int v\,du$

$u = x$ $\quad dv = \sin x\,dx$

$du = dx$ $\quad v = -\cos x$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

$= -x\cos x + \int \cos x\,dx$

$= \boxed{-x\cos x + \sin x + C}$

Going forward, the challenge will be to decide if u-sub or IBP should be used.

u-sub

Do: $\int x\sin x^2\,dx$ $\quad u = x^2 \quad \frac{1}{2}du = x\,dx$

$= \int \sin x^2 \cdot x\,dx$

$= \frac{1}{2}\int \sin u \cdot du$

$= \boxed{-\frac{1}{2}\cos(x^2) + C}$

IBP

Do: $\int x\sin(2x)\,dx$ $\quad u = x \quad dv = \sin 2x\,dx$

$du = dx \quad v = -\frac{1}{2}\cos 2x$

$= uv - \int v\,du$

$= x\left(-\frac{1}{2}\cos 2x\right) + \frac{1}{2}\int \cos 2x\,dx$

$= -\frac{1}{2}x\cos 2x + \frac{1}{2}\cdot\frac{1}{2}\sin 2x + C$

$= \boxed{-\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C}$

$\int \sin(2x)\,dx$

$= \frac{1}{2}\int \sin u\,du$ $\quad u = 2x$

$= -\frac{1}{2}\cos(2x) + C$ $\quad du = 2dx$

$\frac{1}{2}du = dx$

**Needing to Use IBP More Than Once:**

ex. $\int t^2 e^t \, dt = uv - \int v \, du$

$= t^2 e^t - \int e^t \cdot \underbrace{2t \, dt}_{du}$

$= t^2 e^t - 2 \int \underbrace{t}_{u_2} \underbrace{e^t \, dt}_{dv_2}$ ← needs its own IBP

$u_2 v_2 - \int v_2 \, du_2$

$= t^2 e^t - 2 \left( t e^t - \int e^t \, dt \right)$

$= \boxed{t^2 e^t - 2te^t + 2e^t + C}$

$u = t^2 \qquad dv = e^t \, dt$

$du = 2t \, dt \qquad v = e^t$

$u_2 = t \qquad dv_2 = e^t \, dt$

$du_2 = dt \qquad v_2 = e^t$

$\int u \, dv = uv - \int v \, du$

**When power function isn't *u*:**

ex. $\int x^5 \ln x \, dx$

$= \int \underbrace{\ln x}_{u} \cdot \underbrace{x^5 \, dx}_{dv}$

$u = \ln x$

$du = \frac{1}{x} \, dx$

$dv = x^5 \, dx$

$v = \frac{x^6}{6} = \frac{1}{6} \cdot x^6$

$= uv - \int v \, du$

$= \ln x \cdot \frac{x^6}{6} - \frac{1}{6} \int x^6 \cdot \frac{1}{x} \, dx$

$= \frac{1}{6} x^6 \ln x - \frac{1}{6} \int x^5 \, dx$

$= \frac{1}{6} x^6 \ln x - \frac{1}{6} \cdot \frac{x^6}{6} + C$

$= \boxed{\frac{1}{6} x^6 \ln x - \frac{1}{36} x^6 + C}$

$x^6 \cdot \frac{1}{x} = \frac{x^6}{x} = x^5$